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$1/N$ expansion for the Yukawa potential revisited†

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Abstract. In a recent paper, Moreno and Zepeda have applied the $1/N$ expansion to obtain an approximate analytic formula for the ground state energy of a particle bound by a Yukawa potential. Using the method of Mlodinow and Shatz we calculate the energies of the ground state and the first excited state of the system to show that the results provided by the $1/N$ expansion are identical with those obtained by applying either the analytic perturbation theory or the hypervirial equations with the Hellman-Feynman theorem. Also an approximate analytic formula for the ground state energy is proposed.

1. Introduction

Interest in the $1/N$ expansion has continued unabated since the work of Ferrel and Scalapino (1974) on the anharmonic oscillator. The method has drawn considerable attention for the following reasons. Firstly, the expansion, albeit semiclassical, works in many cases extremely well. For the Coulomb potential, in particular, the method looks all the more attractive, for it gives an exact result when summed to infinite order (Mlodinow and Papanicolaou 1980, Van Der Merwe 1983). However, the mathematical foundation of the scheme remains elusive and therefore requires critical investigations (Mlodinow and Papanicolaou 1980, 1981, Papanicolaou 1981). Secondly, the expansion is essentially non-perturbative and hence can possibly provide a satisfactory way of solving the strong coupling problems ('t Hooft 1974a, b, Witten 1979) for which the usual perturbative treatments fail. Besides these reasons, the formalism has a special appeal because it involves only algebraic equations which are easier to handle.

Recently, Moreno and Zepeda (1984, hereafter referred to as MZ) have applied the $1/N$ expansion to the Schrödinger equation with a Yukawa potential. Following a slightly different method to that of Mlodinow and Shatz (1982, hereafter referred to as MS) they have obtained an approximate analytic formula for the ground state energy of the system which is in very good agreement with the numerical results. However, the approximation they have invoked appears (at least to the present author) to be a bit artificial and that their approximation really works, they themselves agree, is rather fortuitous. In the present paper we employ the method of MS in a straightforward manner to calculate the first fourteen terms of the expansion for the ground state energy. We also perform calculations for the first excited state (2s). The main objective is to show that the results which are provided by the $1/N$ expansion are identical with those obtained by using the analytic perturbation theory (McEnnen *et al* 1976) or the

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hypervirial equations with the Hellman-Feynman theorem (Grant and Lai 1979). We also propose an approximate analytic formula for the ground state energy which gives fairly accurate results.

2. The method

2.1. Zeroth-order approximation

The radial part of the N-dimensional Schrödinger equation (in units $m=\hbar=1$) is given by

$$\left[-\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{l(l+N-2)}{2r^2}+V(r)\right]R(r)=ER(r), \tag{1}$$

which on substituting $R(r)=r^{-(N-1)/2}u(r)$ reduces to

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r)+k^2\left(\frac{(1-1/k)(1-3/k)}{8r^2}+\tilde{V}(r)\right)u(r)=Eu(r), \tag{2}$$

where $\tilde{V}(r)=V(r)/k^2$, $k=N+2l$.

Following the method of MZ we write for the Yukawa potential in N dimensions

$$V(r)=-(a/r)\exp[-(9b/N^2)r],$$

which when plugged in (2) leads for $l=0$ to

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r)+k^2\left[\frac{(1-1/k)(1-3/k)}{8r^2}-\frac{\tilde{a}}{r}\exp\left(-\frac{9b}{k^2}r\right)\right]u(r)=Eu(r), \tag{3}$$

where $\tilde{a}=a/k^2$. In the limit of large k ($N\to\infty$), the energy eigenvalue to leading order is given by

$$E_\infty=k^2E^{(-2)}=k^2(1/8r_0^2-\tilde{a}/r_0), \tag{4}$$

where r_0 is to be obtained by minimising the potential $(1/8r^2-\tilde{a}/r)$ which then yields

$$r_0=1/4\tilde{a} \tag{5}$$

and

$$E^{(-2)}=-2\tilde{a}^2. \tag{6}$$

2.2. Higher-order corrections to the ground state energy

Quantum fluctuations around the classical minimum r_0 can be incorporated in the higher-order corrections for which we define $x=r-r_0$. Assuming $u_0(r)=\exp(\phi_0(x))$ for the ground state wavefunction, we obtain from (3)

$$-\tfrac{1}{2}(\phi_0''(x)+\phi_0'^2(x))+k^2V_{\mathrm{eff}}(x)+(-\tfrac{1}{2}k+\tfrac{3}{8})r^{-2}(x)+(9b)\tilde{a}$$
$$-\frac{1}{2!}\frac{(9b)^2}{k^2}\tilde{a}r(x)+\frac{1}{3!}\frac{(9b)^3}{k^4}\tilde{a}r^2(x)-\frac{1}{4!}\frac{(9b)^4}{k^6}\tilde{a}r^3+\ldots=\mathscr{E}_0, \tag{7}$$

where

$$V_{\mathrm{eff}}(x)=1/8r^2(x)-\tilde{a}/r(x)+2\tilde{a}^2, \tag{8}$$

$$\mathscr{E}_0=E_0-k^2E^{(-2)}, \tag{9}$$

E_0 being the ground state energy and $\phi'(x)$ and $\phi''(x)$ represent respectively the first and second derivates of $\phi(x)$ with respect to x.. We next substitute the expansions

$$\mathscr{E}_0 = \sum_{n=-1}^{\infty} E_0^{(n)} k^{-n}, \tag{10}$$

and

$$\phi_0'(x) = \sum_{n=-1}^{\infty} \phi_0^{(n)}(x) k^{-n} \tag{11}$$

in (7) which then reads

$$-\frac{1}{2}\sum_{n=-1}^{\infty} \phi_0^{(n)'}(x)k^{-n} - \frac{1}{2}\sum_{m,n=-1}^{\infty} \phi_0^{(m)}(x)\phi_0^{(n)}(x)k^{-(m+n)} + k^2 V_{\text{eff}}(x)$$
$$+(-\tfrac{1}{2}k+\tfrac{3}{8})r^2(x)+(9b)\tilde{a} - \frac{1}{2!}\frac{(9b)^2}{k^2}\tilde{a}r(x) + \frac{1}{3!}\frac{(9b)^3}{k^4}\tilde{a}r^2(x)$$
$$-\frac{1}{4!}\frac{(9b)^4}{k^6}\tilde{a}r^3(x)+\ldots = \sum_{n=-1}^{\infty} E_0^{(n)}k^{-n}. \tag{12}$$

Now equating the terms of same order in k we obtain the following recurrence relations

$$\phi_0^{(-1)}(x) = -(2V_{\text{eff}}(x))^{1/2}, \tag{13a}$$

$$(2V_{\text{eff}}(x))^{1/2}\phi_0^{(0)}(x) = E_0^{(-1)} + \tfrac{1}{2}r^{-2}(x) + \tfrac{1}{2}\phi_0^{(-1)'}(x), \tag{13b}$$

$$(2V_{\text{eff}}(x))^{1/2}\phi_0^{(1)}(x) = E_0^{(0)} - \tfrac{3}{8}r^{-2}(x) + \tfrac{1}{2}\phi_0^{(0)'}(x) + \tfrac{1}{2}\phi_0^{(0)^2}(x) - 9b\tilde{a}, \tag{13c}$$

$$(2V_{\text{eff}}(x))^{1/2}\phi_0^{(n)}(x)$$
$$= E_0^{(n-1)} + \tfrac{1}{2}\phi_0^{(n-1)'}(x) + \tfrac{1}{2}\sum_{m=0}^{n} \phi_0^{(m)}(x)\phi_0^{(n-m-1)}(x)$$
$$+(-1)^{(n+1)/2}\frac{(9b)^{(n+1)/2}}{[(n+1)/2]!}\tilde{a}r^{(n-1)/2}(x)\Bigg|_{=0 \text{ when } n \text{ is even}}; \qquad n>1. \tag{13d}$$

Since the effective potential $V_{\text{eff}}(x)$ vanishes at the minimum ($r = r_0$), we have for the higher-order corrections to the ground state energy

$$E_0^{(-1)} = -\tfrac{1}{2}r_0^{-2} - \tfrac{1}{2}\phi_0^{(-1)'}(0), \tag{14a}$$

$$E_0^{(0)} = \tfrac{3}{8}r_0^{-2} - \tfrac{1}{2}\phi_0^{(0)'}(0) - \tfrac{1}{2}\phi_0^{(0)^2}(0) + (9b)\tilde{a}, \tag{14b}$$

$$E_0^{(n)} = -\tfrac{1}{2}\phi_0^{(n)'}(0) - \tfrac{1}{2}\sum_{m=0}^{n}\phi_0^{(m)}(0)\phi_0^{(n-m)}(0)$$
$$+(-1)^{n/2}\frac{(9b)^{n/2+1}}{(n/2+1)!}\tilde{a}r_0^{n/2}\Bigg|_{=0 \text{ when } n \text{ is odd}}; \qquad n>0. \tag{14c}$$

2.3. Higher-order corrections for the first excited state (2s)

Defining as before $x = r - r_0$ and assuming that the wavefunction for the first excited state is of the form

$$u_1(r) = (x-c)\exp(\phi_1(x)),$$

we obtain from (3)

$$-\tfrac{1}{2}(\phi_1''(x)+\phi_1'^2(x))(x-c)-\phi_1'(x)+[k^2V_{\text{eff}}(x)+(-\tfrac{1}{2}k+\tfrac{3}{8})r^{-2}(x)](x-c)$$
$$+\left((9b)\tilde{a}-\frac{1}{2!}\frac{(9b)^2}{k^2}\tilde{a}r(x)+\frac{1}{3!}\frac{(9b)^3}{k^4}\tilde{a}r^2(x)-\frac{1}{4!}\frac{(9b)^4}{k^6}\tilde{a}r^3(x)+\dots\right)(x-c)=(x-c)\mathscr{E}_1, \tag{15}$$

where $V_{\text{eff}}(x)$ is given by (8), and

$$\mathscr{E}_1=E_1-k^2E^{(-2)}, \tag{16}$$

E_1 being the energy of the first excited state. We now make the following expansions

$$\phi_1'(x)=\sum_{n=-1}^{\infty}\phi_1^{(n)}(x)k^{-n}, \tag{17}$$

$$\mathscr{E}_1=\sum_{n=-1}^{\infty}E_1^{(n)}k^{-n}, \tag{18}$$

and

$$c=\sum_{n=1}^{\infty}C^{(n)}k^{-n}, \tag{19}$$

and substitute these in (15) to get

$$\left(-\tfrac{1}{2}\sum_{n=-1}^{\infty}\phi_1^{(n)'}(x)k^{-n}-\tfrac{1}{2}\sum_{m,n=-1}^{\infty}\phi_1^{(m)}(x)\phi_1^{(n)}(x)k^{-(m+n)}\right)\left(x-\sum_{n=1}^{\infty}C^{(n)}k^{-n}\right)$$
$$-\sum_{n=-1}^{\infty}\phi_1^{(n)}(x)k^{-n}+k^2V_{\text{eff}}(x)\left(x-\sum_{n=1}^{\infty}C^{(n)}k^{-n}\right)$$
$$+\left((9b)\tilde{a}-\frac{1}{2!}\frac{(9b)^2}{k^2}\tilde{a}r(x)+\frac{1}{3!}\frac{(9b)^3}{k^4}\tilde{a}r^2(x)-\frac{1}{4!}\frac{(9b)^4}{k^6}\tilde{a}r^3(x)+\dots\right)$$
$$\times\left(x-\sum_{n=1}^{\infty}C^{(n)}k^{-n}\right)$$
$$=\left(x-\sum_{n=1}^{\infty}C^{(n)}k^{-n}\right)\sum_{n=-1}^{\infty}E_1^{(n)}k^{-n}. \tag{20}$$

Again equating the terms of same order in k we generate a set of recurrence relations

$$\phi_1^{(-1)}(x)=-(2V_{\text{eff}}(x))^{1/2}, \tag{21a}$$

$$x(2V_{\text{eff}}(x))^{1/2}\phi_1^{(0)}(x)=x(E_1^{(-1)}+\tfrac{1}{2}\phi_1^{(-1)'}(x)+\tfrac{1}{2}r^{-2}(x))+\phi_1^{(-1)}(x), \tag{21b}$$

$$x(2V_{\text{eff}}(x))^{1/2}\phi_1^{(1)}(x)$$
$$=x(E_1^{(0)}+\tfrac{1}{2}\phi_1^{(0)'}(x)+\tfrac{1}{2}\phi_1^{(0)^2}(x)-\tfrac{3}{8}r^{-2}(x)-(9b)\tilde{a})-C^{(1)}(E_1^{(-1)}$$
$$+\tfrac{1}{2}\phi_1^{(-1)'}(x)+\phi_1^{(-1)}(x)\phi_1^{(0)}(x)+\tfrac{1}{2}r^{-2}(x))+\phi_1^{(0)}(x), \tag{21c}$$

$$x(2V_{\rm eff}(x))^{1/2}\phi_1^{(2)}(x)$$
$$= x(E_1^{(1)}+\tfrac{1}{2}\phi_1^{(1)'}(x)+\phi_1^{(1)}(x)\phi_1^{(0)}(x)) - C^{(1)}(E_1^{(0)}+\tfrac{1}{2}\phi_1^{(0)'}(x)$$
$$+\phi_1^{(1)}(x)\phi_1^{(-1)}(x)+\tfrac{1}{2}\phi_1^{(0)^2}(x)-9b\tilde{a}-\tfrac{3}{8}r^{-2}(x)) - C^{(2)}(E_1^{(-1)}$$
$$+\tfrac{1}{2}\phi_1^{(-1)'}(x)+\phi_1^{(0)}(x)\phi_1^{(-1)}(x)+\tfrac{1}{2}r^{-2}(x))+\phi_1^{(1)}(x), \qquad (21d)$$

$$x(2V_{\rm eff}(x))^{1/2}\phi_1^{(n)}(x)$$
$$= x\Bigg(E_1^{(n-1)}+\tfrac{1}{2}\phi_1^{(n-1)'}(x)+\tfrac{1}{2}\sum_{m=0}^{n-1}\phi_1^{(m)}(x)\phi_1^{(n-m-1)}(x)$$
$$+(-1)^{(n+1)/2}\frac{(9b)^{n+1/2}}{[(n+1)/2]!}\tilde{a}r^{(n-1)/2}(x)\Bigg|_{=0\text{ when } n \text{ is even}}\Bigg)$$
$$-\sum_{m=1}^{n-2}C^{(m)}\Bigg(E_1^{(n-m-1)}+\tfrac{1}{2}\phi_1^{(n-m-1)'}(x)+\tfrac{1}{2}\sum_{p=-1}^{n-m}\phi_1^{(p)}(x)$$
$$\times\phi_1^{(n-m-p-1)}(x)+(-1)^{(n-m+1)/2}\frac{(9b)^{(n-m+1)/2}}{[(n-m+1)/2]!}$$
$$\times\tilde{a}r^{(n-m-1)/2}(x)\Bigg|_{\substack{=0\text{ when } m \text{ and } n \text{ are}\\ \text{both even or both odd}}}\Bigg)$$
$$-C^{(n-1)}(E_1^{(0)}+\tfrac{1}{2}\phi_1^{(0)'}(x)+\phi_1^{(1)}(x)\phi_1^{(-1)}(x)+\tfrac{1}{2}\phi_1^{(0)^2}(x)$$
$$-(9b)\tilde{a}-\tfrac{3}{8}r^{-2}(x)) - C^{(n)}(E_1^{(-1)}+\tfrac{1}{2}\phi_1^{(-1)'}(x)+\phi_1^{(-1)}(x)\phi_1^{(0)}(x)$$
$$+\tfrac{1}{2}r^{-2}(x))+\phi_1^{(n-1)}(x). \qquad (21e)$$

The above equations are now to be solved to obtain the higher-order corrections to the energy of the first excited state.

3. Results and discussions

3.1. The ground state

Solving (13) and (14), we obtain

$$\phi_0^{(-1)}(x) = -2\tilde{a}+1/2r(x), \qquad \phi_0^{(0)}(x) = -2\tilde{a}-1/2r(x),$$
$$\phi_0^{(1)}(x) = -2\tilde{a}, \qquad \phi_0^{(2)}(x) = -2\tilde{a},$$
$$\phi_0^{(3)} = -2\tilde{a}+[(9b)^2/4]r(x), \qquad \phi_0^{(4)}(x) = -2\tilde{a}-[(9b)^2/4]r(x),$$
$$\phi_0^{(5)}(x) = -2\tilde{a}-[(9b)^3/48\tilde{a}]r(x)-[(9b)^3/12]r^2(x),$$
$$\phi_0^{(6)}(x) = -2\tilde{a}+[(9b)^3/48\tilde{a}]r(x)+[(9b)^3/12]r^2(x),$$
$$\phi_0^{(7)}(x) = -2\tilde{a}+[(9b)^3/48\tilde{a}]r(x)+[(9b)^4/192\tilde{a}^2]r(x)$$
$$+[(9b)^4/48\tilde{a}]r^2(x)+[(9b)^4/48]r^3(x),$$
$$\phi_0^{(8)}(x) = -2\tilde{a}-[(9b)^3/48\tilde{a}]r(x)-[3(9b)^4/256\tilde{a}^2]r(x)$$
$$-[3(9b)^4/64\tilde{a}]r^2(x)-[(9b)^4/48]r^3(x),$$

$$\phi_0^{(9)}(x) = -2\tilde{a} + [(9b)^4/384\tilde{a}^2]r(x) + [(9b)^4/32\tilde{a}]r^2(x) - [7(9b)^5/5120\tilde{a}^3]r(x) - [7(9b)^5/1280\tilde{a}^2]r^2(x) - [11(9b)^5/960\tilde{a}]r^3(x) - [(9b)^5/240]r^4(x), \tag{22}$$

$$\phi_0^{(10)}(x) = -2\tilde{a} + [(9b)^4/96\tilde{a}]r(x) - [(9b)^4/192\tilde{a}]r^2(x) + [(9b)^5/320\tilde{a}^3]r(x) + [(9b)^5/80\tilde{a}^2]r^2(x) + [29(9b)^5/960\tilde{a}]r^3(x) + [(9b)^5/240]r^4(x),$$

$$\phi_0^{(11)}(x) = -2\tilde{a} - [(9b)^4/128\tilde{a}^2]r(x) + [9(9b)^5/5120\tilde{a}^3]r(x) + [(9b)^5/640\tilde{a}^2]r^2(x) - [5(9b)^5/192\tilde{a}]r^3(x) + [11(9b)^6/23040\tilde{a}^4]r(x) + [11(9b)^6/5760\tilde{a}^3]r^2(x) + [53(9b)^6/11520\tilde{a}^2]r^3(x) + [13(9b)^6/2880\tilde{a}]r^4(x) + [(9b)^6/1440]r^5(x),$$

$$\phi_0^{(12)}(x) = -2\tilde{a} + [(9b)^4/768\tilde{a}^2]r(x) - [3(9b)^5/320\tilde{a}^3]r(x) - [(9b)^5/40\tilde{a}^2]r^2(x) + [7(9b)^5/960\tilde{a}]r^3(x) - [103(9b)^3/73728\tilde{a}^4]r(x) - [103(9b)^6/18432\tilde{a}^3]r^2(x) - [7(9b)^6/480\tilde{a}^2]r^3(x) - [73(9b)^6/5760\tilde{a}]r^4(x) - [(9b)^6/1440]r^5(x),$$

and

$$E_0^{(-1)} = -4\tilde{a}^2, \qquad E_0^{(0)} = -6\tilde{a}^2 + (9b)\tilde{a}, \qquad E_0^{(1)} = -8\tilde{a}^2,$$
$$E_0^{(2)} = -10\tilde{a}^2 - (9b)^2/8, \qquad E_0^{(3)} = -12\tilde{a}^2 + (9b)^2/8,$$
$$E_0^{(4)} = -14\tilde{a}^2 + (9b)^3/96\tilde{a}, \qquad E_0^{(5)} = -16\tilde{a}^2 - (9b)^3/96\tilde{a},$$
$$E_0^{(6)} = -18a^2 - (9b)^3/96\tilde{a} - (9b)^4/384\tilde{a}^2,$$
$$E_0^{(7)} = -20\tilde{a}^2 + (9b)^3/96\tilde{a} + 3(9b)^4/512\tilde{a}^2,$$
$$E_0^{(8)} = -22\tilde{a}^2 - (9b)^4/768\tilde{a}^2 + 7(9b)^5/10240\tilde{a}^3, \tag{23}$$
$$E_0^{(9)} = -24\tilde{a}^2 - (9b)^4/192\tilde{a}^2 - (9b)^5/640\tilde{a}^3,$$
$$E_0^{(10)} = -26\tilde{a}^2 + (9b)^4/256\tilde{a}^2 - 9(9b)^5/10240\tilde{a}^3 - 11(9b)^6/46080\tilde{a}^4,$$
$$E_0^{(11)} = -28\tilde{a}^2 - (9b)^4/1536\tilde{a}^2 + 3(9b)^5/640\tilde{a}^3 + 103(9b)^6/147456\tilde{a}^4,$$
$$E_0^{(12)} = -30\tilde{a}^2 + \text{terms of higher order than } (9b)^4.$$

The ground state energy is now given by

$$E_0 = k^2E^{(-2)} + \sum_{n=-1}^{\infty} k^{-n}E_0^{(n)} \tag{24}$$

$$= -\frac{2a^2}{(k-1)^2} + \frac{4a^2}{k^2}\left(\frac{9b}{4a}\right) - \frac{2a^2}{k^2}\left(1-\frac{1}{k}\right)\left(\frac{9b}{4a}\right)^2 + \frac{2a^2}{3k^2}\left(1-\frac{1}{k}-\frac{1}{k^2}+\frac{1}{k^3}\right)\left(\frac{9b}{4a}\right)^3 - \frac{2a^2}{3k^2}\left(1-\frac{9}{4k}+\frac{1}{2k^2}+\frac{2}{k^3}-\frac{1}{2k^4}+\frac{1}{4k^5}\right)\left(\frac{9b}{4a}\right)^4 + \frac{7a^2}{10k^2}\left(1-\frac{16}{7k}-\frac{9}{7k^2}+\frac{48}{7k^3}+\dots\right)\left(\frac{9b}{4a}\right)^5 - \frac{44a^2}{45k^2}\left(1-\frac{515}{176k}+\dots\right)\left(\frac{9b}{4a}\right)^6 + \dots . \tag{24a}$$

Putting $9b/4a=\beta$ and $k=3$ in (24a) we finally obtain

$$E_0/a^2=-\tfrac{1}{2}+\tfrac{4}{9}\beta-\tfrac{4}{27}\beta^2+\tfrac{32}{729}\beta^3-\tfrac{176}{6561}\beta^4+\ldots. \tag{25}$$

It is interesting to note that our result to this order is identical with that obtained earlier by Grant and Lai (1979a) by applying the hypervirial equations with the Hellman–Feynman theorem. The first four terms of (25) have also been obtained by McEnnen *et al* (1976) by using the analytic perturbation theory. It has also been shown by Lai (1979) that one can obtain the first four terms with the help of the Hellman–Feynman theorem alone.

We now approximate the expression (25) by an infinite geometric series to obtain an analytic formula for the ground state energy of the system

$$E_0/a^2=\tfrac{5}{6}-4/(\beta+3). \tag{26}$$

To show that the above approximate formula works well we compute

$$\varepsilon=-\tfrac{81}{8}E_0/a^2 \tag{26a}$$

for different values of β in the range $0\leq\beta\leq 1$ and then compare our results with those of MZ (table 1). MZ have also shown that as one goes on increasing the number of terms while calculating the energy from (24a), the result tends to deviate more and more from the actual numerical value when β approaches 1. This apparently raises a doubt about the very applicability of the expansion. However, when reckoned as a series in β this ticklish problem disappears, which suggests that though the method is non-perturbative in the sense that it is not an expansion in the coupling constant, for the case of the Yukawa potential, β turns out to be a good perturbation parameter. This can be made evident by truncating the series for energy at β^4, which then yields fairly accurate results for $0\leq\beta<1$ (table 1).

Table 1. Comparison of the values of ε for $0\leq\beta\leq 1$ as calculated from the approximate analytic formula of MZ (ε_{MZ}) with those of the present work ($\varepsilon_{present}$). Also the values of ε to order β^4 are presented.

β	ε_{MZ}	$\varepsilon_{present}$	ε (to order β^4)
0.1	4.6270	4.6276	4.6271
0.2	4.2187	4.2188	4.2194
0.3	3.8350	3.8352	3.8377
0.4	3.4735	3.4743	3.4810
0.5	3.1321	3.1339	3.1489
0.6	2.8269	2.8125	2.8417
0.7	2.5026	2.5084	2.5603
0.8	2.2105	2.2204	2.3062
0.9	1.9322	1.9471	2.0817
1.0	1.6661	1.6875	

3.2. The first excited state

Equations (21) can be solved for $\phi_1^{(n)}$, $C^{(n)}$ and $E_1^{(n)}$ to give

$\phi_1^{(-1)}(x)=-2\tilde{a}+1/2r(x),\qquad \phi_1^{(0)}(x)=2\tilde{a}-1/2r(x),$

$\phi_1^{(1)}(x)=-2\tilde{a},\qquad \phi_1^{(2)}(x)=2\tilde{a},\qquad \phi_1^{(3)}(x)=-2\tilde{a}+[(9b)^2/4]r(x),$

$$\phi_1^{(4)}(x)=2\tilde{a}-(9b)^2/8\tilde{a}+[(9b)^2/4]r(x),$$

$$\phi_1^{(5)}(x)=-2\tilde{a}-[(9b)^3/48\tilde{a}]r(x)-[(9b)^3/12]r^2(x),$$

$$\phi_1^{(6)}(x)=2\tilde{a}+(9b)^2/8\tilde{a}+(9b)^3/48\tilde{a}^2-[5(9b)^3/48\tilde{a}]r(x)-[(9b)^3/12]r^2(x), \qquad (27a)$$

$$\phi_1^{(7)}(x)=-2\tilde{a}+(9b)^3/24\tilde{a}^2-[7(9b)^3/48\tilde{a}]r(x)+[(9b)^4/192\tilde{a}^2]r(x)$$
$$+[(9b)^4/48\tilde{a}]r^2(x)+[(9b)^4/48]r^3(x),$$

$$\phi_1^{(8)}(x)=2\tilde{a}-[(9b)^3/16\tilde{a}]r(x)-5(9b)^4/512\tilde{a}^3+[7(9b)^4/256\tilde{a}^2]r(x)$$
$$+[5(9b)^4/64\tilde{a}]r^2(x)+[(9b)^4/48]r^3(x);$$

$$C^{(1)}=0, \qquad C^{(2)}=-1/4\tilde{a}, \qquad C^{(3)}=0,$$
$$C^{(4)}=(9b)^2/128\tilde{a}^3, \qquad C^{(5)}=0,$$
$$C^{(6)}=-3(9b)^2/128\tilde{a}^3-(9b)^3/768\tilde{a}^4, \qquad C^{(7)}=-(9b)^3/384\tilde{a}^4, \qquad (27b)$$
$$C^{(8)}=3(9b)^2/128\tilde{a}^3+(9b)^3/384\tilde{a}^4+7(9b)^4/8192\tilde{a}^5,$$
$$C^{(9)}=(9b)^3/128\tilde{a}^4+7(9b)^4/6144\tilde{a}^5,$$

$$E_1^{(-1)}=4\tilde{a}^2, \qquad E_1^{(0)}=-6\tilde{a}^2+9b\tilde{a}, \qquad E_1^{(1)}=8\tilde{a}^2,$$
$$E_1^{(2)}=-10\tilde{a}^2-(9b)^2/8, \qquad E_1^{(3)}=12\tilde{a}^2-5(9b)^2/8,$$
$$E_1^{(4)}=-14\tilde{a}^2+(9b)^3/96\tilde{a}, \qquad E_1^{(5)}=16\tilde{a}^2+13(9b)^3/96\tilde{a}, \qquad (27c)$$
$$E_1^{(6)}=-18\tilde{a}^2+23(9b)^3/96\tilde{a}-(9b)^4/384\tilde{a}^2,$$
$$E_1^{(7)}=20\tilde{a}^2+[11(9b)^3/96\tilde{a}]-23(9b)^4/512\tilde{a}^2,$$
$$E_1^{(8)}=-22\tilde{a}^2-91(9b)^4/768\tilde{a}^2+7(9b)^5/10240\tilde{a}^3.$$

The energy of the first excited state is now given by

$$E_1=k^2E^{(-2)}+\sum_{n=-1}^{\infty}k^{-n}E_1^{(n)}$$
$$=-\frac{2a^2}{(k+1)^2}+\frac{4a^2}{k^2}\beta-\frac{2a^2}{k^2}\left(1+\frac{5}{k}\right)\beta^2+\frac{2a^2}{3k^2}\left(1+\frac{13}{k}+\frac{23}{k^2}+\frac{11}{k^3}\right)\beta^3$$
$$-\frac{2a^2}{3k^2}\left(1+\frac{69}{4k}+\frac{91}{2k^2}+\ldots\right)\beta^4+\frac{7a^2}{10k^2}(1+\ldots)\beta^5-\ldots. \qquad (28)$$

For $N=3$ which is the dimension of interest, we finally obtain

$$E_1/a^2=-\tfrac{1}{8}+\tfrac{4}{9}\beta-\tfrac{16}{27}\beta^2+\tfrac{448}{729}\beta^3+\ldots. \qquad (29)$$

It is immediately seen that the above expression reproduces the proper Coulomb limit. We again emphasise that (29) to order β^3 is identical with the earlier results (McEnnen *et al* 1976, Grant and Lai 1979, Lai 1979).

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